Local Stability of Semidefinite Relaxations

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Overview

Goals

1. Examine SDP relaxations of QCQPs closely
2. Analyze when the relaxation is exact
   - Exactness with noisy parameters
3. (If time), examine applications in statistical estimation

Setting

Many problems in statistics and engineering are of the form

$$q^\star = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad x^\top Q x$$
$$\text{subject to} \quad q_i(x) = 0, \quad i = 1, 2, \ldots, k \qquad \text{(Q)}$$

We define $g \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ as the corresponding system of quadratic polynomials. In general, this problem is not convex, since the set of feasible points, $\mathcal{X}$, is not convex, and if Q is not positive semidefinite, the cost is not convex.

Dual of a QCQP

However, let's look at the dual of (Q) more explicitly. We can rewrite any quadratic constaint as

$$
\begin{aligned}
q_i(x) = x^\top A_i x + 2b_i^\top x + c_i &= \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x \\ 1 \end{pmatrix}^\top Q_i \begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \left\langle \begin{bmatrix} xx^\top & x \\ x^\top & 1 \end{bmatrix}, Q_i \right\rangle .
\end{aligned}
$$

This immediately gives a reformulation of our original problem:

$$p^\star = \underset{x \in \mathbb{R}^n, X \in \mathbb{S}^n}{\text{minimize}} \quad \left\langle \begin{bmatrix} X & x \\ x^\top & 1 \end{bmatrix}, Q \right\rangle$$

$$\text{subject to} \quad \left\langle \begin{bmatrix} X & x \\ x^\top & 1 \end{bmatrix}, Q_i \right\rangle = 0, \quad i = 1, 2, \dots, k,$$

$$\begin{bmatrix} X & x \\ x^\top & 1 \end{bmatrix} \succeq 0 \qquad \text{(P)}$$

Note that we dropped the rank constraint to obtain an SDP relaxation. This is known as Shor's relaxation. Note that this also let's us assume that all quadratics are homogeneous.

Let us now examine the dual of the problem, which is

$$d^\star = \underset{\lambda \in \mathbb{R}^k}{\text{maximize}} \quad \langle \lambda, c \rangle$$
$$\text{subject to} \quad Q + \sum_{i=1}^{k} \lambda_i Q_i \succeq 0 \qquad \text{(D)}$$

A natural question is: when do we have zero duality gap? Note that we already have

$$q^\star \geq p^\star \geq d^\star$$

by the relaxation and weak duality.

Zero Duality Gap

Suppose we have a homogeneous QCQP ($b_i = 0$ for all i) and that $c_i \neq 0$ for some i. Then, define $H(\lambda) = Q + \sum_{i=1}^{k} \lambda_i Q_i$. Note that this is $\frac{1}{2}\nabla_{xx} L(x, \lambda)$, half the Hessian of the Lagrangian. We say λ is a **Lagrange multiplier** of x if the following equivalent conditions hold:

$$\nabla_x L(x, \lambda) = 0$$
$$H(\lambda)x = 0.$$

Let $\Lambda(x)$ be the set of Lagrange multipliers at x. Then $\Lambda(x)$ is an affine set, and we will say x is a **critical point** of (Q) if $\Lambda(x)$ is nonempty and x is feasible.

Strong Duality

Theorem ([1, Lemma 2.1])

Suppose that there exists $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$ such that

1. $q_i(x) = 0$ *for all* i *(primal feasibility)*
2. $H(\lambda) \succeq 0$ *(dual feasibility)*
3. $H(\lambda)x = 0$ *(λ is a Lagrange multiplier).*

Then x is optimal for (Q)*, λ is optimal for* (D)*, and $q^\star = d^\star$. Moreover, if $\operatorname{rank}(H(\lambda)) = n-1$, then $xx^\top$ is the unique minimizer of* (P) *and x is the unique minimizer of* (Q) *(up to sign).*

Proof

Proof.

Since $H(\lambda)x = 0$ and x is feasible,

$$0 = x^\top H(\lambda)x = x^\top Qx + \sum_{i=1}^{k} \lambda_i x^\top A_i x = x^\top Qx - \sum_{i=1}^{k} \lambda_i c_i$$

so therefore $q^\star = d^\star$.

Suppose S is an optimal solution of (P). Then, $S \neq 0$ if at least one c_i is nonzero. By complementary slackness, $\langle H(\lambda), S\rangle = 0$, and since both are PSD, $\mathrm{rank}(H(\lambda)) + \mathrm{rank}(S) \leq n$. If $\mathrm{rank}(H(\lambda)) = n - 1$, then $\mathrm{rank}(S) = 1$. This also implies that all solutions S have rank 1, so there cannot be another distinct solution. Otherwise, their convex hull would have a solution of rank 2. Therefore, we can recover a solution for (Q), up to a choice of sign.

We want to study what happens when we perturb the problem data slightly. Most importantly, will exactness and/or strong duality be preserved? For simplicity, suppose we have $\Theta \subseteq \mathbb{R}^d$ and continuous functions $(Q_i(\theta), c_i(\theta)) \colon \Theta \to \mathbb{S}^n_+ \times \mathbb{R}$. Thus, we have

$$q^\star(\theta) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad x^\top Q(\theta) x \quad \text{subject to} \quad q_i(x,\theta) = 0, \quad i = 1, 2, \ldots, k \tag{Q_θ}$$

$$p^\star(\theta) = \underset{X \in \mathbb{S}^n}{\text{minimize}} \quad \langle X, Q(\theta) \rangle \quad \text{subject to} \quad \langle X, Q_i(\theta) \rangle = c_i(\theta), \quad i = 1, 2, \ldots, k, \quad X \succeq 0 \tag{P_θ}$$

$$d^\star(\theta) = \underset{\lambda \in \mathbb{R}^k}{\text{maximize}} \quad \langle \lambda, c(\theta) \rangle \quad \text{subject to} \quad Q(\theta) + \sum_{i=1}^{k} \lambda_i Q_i(\theta) \succeq 0 \tag{D_θ}$$

SDP Stability

Definition

We say that (Q_θ) is **SDP stable near** $\overline{\theta}$ if there exists $\varepsilon > 0$ such that $q^\star(\theta) = p^\star(\theta) = d^\star(\theta)$ when $\|\theta - \overline{\theta}\| < \varepsilon$.

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We will need some additional regularity assumptions to show stability.

Regularity Assumptions

Definition

Given $g\colon \mathbb{R}^n \to \mathbb{R}^k$, let $\mathcal{X} = \{x \in \mathbb{R}^n : g(x) = 0\}$. The **Abadie constraint qualification** ($\mathrm{ACQ}_{\mathcal{X}}(x)$) holds at $x \in \mathcal{X}$ if $\mathcal{X}$ is a smooth manifold near x and $\operatorname{rank}(D_g(x)) = \operatorname{codim}(\mathcal{X}) =: n - \dim(\mathcal{X})$, where D_g is the Jacobian matrix of g.

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- This intuitively provides some regularity of g.
- The smoothness and rank conditions suggests that g does not define a degenerate feasible region.
- $\mathrm{ACQ}_{\mathcal{X}}(x)$ guarantees the existence of Lagrange multipliers at x, that $\Lambda(x) \neq \varnothing$.
- In the case where g is a polynomial mapping, $\mathrm{ACQ}_{\mathcal{X}}(x)$ holds if and only if $\operatorname{rank}(D_g(x)) = \operatorname{codim}(\mathcal{X})$.

Robustness of Perturbations

For the following theorems, we assume that all parameters other than the cost are constant with respect to θ.

Theorem

Consider (Q_θ). *Suppose* $Q(\theta)$ *is continuous, and* $c \neq 0$. *Suppose* $\overline{\theta}$ *is such that* $Q(\overline{\theta}) \succeq 0$ *has corank 1, and* $q^\star(\overline{\theta}) = 0$. *If* $ACQ_{\mathcal{X}}(\overline{x})$ *holds, then* (Q_θ) *is SDP stable near* $\overline{\theta}$ *and* (P_θ) *shares its minimizer.*

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Proving this result requires some preliminary results that are of their own interest.

Lagrange Multiplier Mapping

Definition

The Lagrange Multiplier Mapping is $\mathfrak{L}\colon \Theta \rightrightarrows \mathbb{R}^n \times \mathbb{R}^k$ where

$$\mathfrak{L}(\theta) = \{(x, \lambda) : x \text{ is feasible for } (\mathrm{Q}_\theta), \lambda \in \Lambda_\theta(x)\}\,.$$

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Definition

We say $\mathfrak{L}$ is **weakly continuous** at $\bar{\ell} = (\bar{x}, \bar{\lambda}) \in \mathfrak{L}(\bar{\theta})$ if there exists $\ell_\theta \in \mathfrak{L}(\theta)$ such that $\ell_\theta \to \bar{\ell}$ as $\theta \to \bar{\theta}$.

Applying Weak Continuity

Theorem

Suppose $\overline{\theta}$ *is such that* (Q_θ) *has zero duality gap, and let* $(\overline{x}, \overline{\lambda})$ *is a primal-dual optimal pair for* (Q_θ). *If* $H_{\overline{\theta}}(\overline{\lambda})$ *has corank 1 and* $\mathfrak{L}$ *is weakly continuous at* $(\overline{x}, \overline{\lambda})$, *then* (Q_θ) *is SDP stable at* $\overline{\theta}$ *and* (P_θ) *shares its minimizer.*

Applying Weak Continuity

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Suppose $\overline{\theta}$ is such that (Q_{θ}) has zero duality gap, and let $(\overline{x}, \overline{\lambda})$ is a primal-dual optimal pair for (Q_{θ}). If $H_{\overline{\theta}}(\overline{\lambda})$ has corank 1 and $\mathfrak{L}$ is weakly continuous at $(\overline{x}, \overline{\lambda})$, then (Q_{θ}) is SDP stable at $\overline{\theta}$ and (P_{θ}) shares its minimizer.

Proof.

By weak continuity, there exists $(x_\theta, \lambda_\theta)$ such that x_θ is feasible, $\lambda_\theta \in \Lambda_\theta(x_\theta)$, and $(x_\theta, \lambda_\theta) \to (\overline{x}, \overline{\lambda})$ as $\theta \to \overline{\theta}$. Then, it follows that $H_\theta(\lambda_\theta) \to H_{\overline{\theta}}(\overline{\lambda})$, since we assume that the constraints and cost vary continuously with θ.

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Since $H_\theta(\lambda_\theta) \succeq 0$, and $H_{\overline{\theta}}(\overline{\lambda})$ has corank 1, $H_{\overline{\theta}}(\overline{\lambda})$ has $n-1$ positive eigenvalues. Because $H_\theta(\lambda_\theta)x_\theta = 0$, $H_\theta(\lambda_\theta)$ has at most $n-1$ positive eigenvalues. By continuity of eigenvalues, as $\theta \to \overline{\theta}$, $H_\theta(\lambda_\theta)$ has $n-1$ positive eigenvalues. Applying Theorem 1, we complete the proof.

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- We have shown that given the weak continuity, we have SDP stability.
- We need to analyze when weak continuity holds.
- We are want to know how $q^\star(\theta)$ behaves as θ varies.

Lemma

Let $F\colon S \times \theta \to \mathbb{R}$ *be a continuous function with* $S \subseteq \mathbb{R}^n$ *compact. Then* $\theta \mapsto \min_{x \in S} F(x, \theta)$ *is continuous.*

Sketch of Proof.
The essential fact is that $F(\cdot, \theta)$ is uniformly continuous since S is compact.

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Sketch of Proof.
The essential fact is that $F(\cdot, \theta)$ is uniformly continuous since S is compact.

- We will use this fact to prove the local continuity of the optimal solution.

Continuity of Primal Optimal Solution

Theorem

Let $x_\theta^\star$ be an optimal solution of (Q_θ). Then $x_\theta^\star \to x_{\overline{\theta}}^\star$ as $\theta \to \overline{\theta}$.

Proof.

Let $x = (x_1, y)$ where $y = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Since at least one $c_i \neq 0$, let us assume $c_1 \neq 0$. Because $F_{\overline{\theta}}$ has corank 1, we may assume after a change of coordinates that $\overline{x} = (1, 0)$ and the cost is $\|y\|_2^2$. Our first goal will be to bound the size of the feasible set.

Proof Cont.

Proof.

Note that since we assumed homogeneous polynomials, we have for any feasible x

$$q_1(x) = x^\top Q_1 x - 1 = 0.$$

For $\overline{x} = (1, 0)$, this means that the top left entry of Q_1 must be 1. Thus, we may write

$$q_1(x) = (x_1 - v^\top y)^2 - (y^\top V y + 1) = 0,$$

so $x_1 = v^\top y \pm \sqrt{y^\top V y + 1}$. Thus,

$$|x_1| \leq (\|v\| + \|V\|_2^{1/2})\|y\| + 1 \leq \underbrace{(1 + \|v\| + \|V\|_2^{1/2})}_{\alpha}(\|y\| + 1)$$

Proof Cont.

Assume θ is close enough to $\overline{\theta}$ so that $\|Q_\theta - Q_{\overline{\theta}}\|_2 < \frac{1}{8\alpha^2}$. Now suppose that $x = (x_1, y)$ has $\|y\| > 1$ and is feasible. Then,

$$\begin{aligned} q_\theta(x) \geq q_{\overline{\theta}}(x) - |q_{\overline{\theta}}(x) - q_\theta(x)| &\geq \|y\|^2 - \|Q_\theta - Q_{\overline{\theta}}\|_2 \|x\|^2 \\ &\geq \|y\|^2 - \frac{1}{8\alpha^2}\alpha^2(\|y\| + 1)^2 \\ &\geq (1 - \frac{1}{8} \cdot 4)\|y\|^2 \\ &\geq \frac{1}{2}. \end{aligned}$$

Proof Cont.

Assume θ is close enough to $\overline{\theta}$ so that $\|Q_\theta - Q_{\overline{\theta}}\|_2 < \frac{1}{8\alpha^2}$. Now suppose that $x = (x_1, y)$ has $\|y\| > 1$ and is feasible. Then,

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Proof Cont.

We can also see that $\overline{x}$ has a lower cost:

$$q_\theta(\overline{x}) \leq q_{\overline{\theta}}(x) + |q_{\overline{\theta}}(x) - q_\theta(x)| \leq \|Q_\theta - Q_{\overline{\theta}}\|_2 \|\overline{x}\|^2 \leq \frac{1}{8\alpha^2}.$$

Thus, all compact solutions must belong to the compact set $S = \{x = (x_1, y) : \|y\| \leq 1, \|x\| \leq 2\alpha\}$. Applying Theorem 8, we see that

$$\|y_\theta^\star\|^2 \leq |q_\theta(x_\theta^\star)| + |q_{\overline{\theta}}(x_\theta^\star) - q_\theta(x_\theta^\star)| \leq \|q_\theta - q_{\overline{\theta}}\|_{\sup} + |q_\theta(x_\theta^\star)| \to 0,$$

as $\theta \to \overline{\theta}$. Since a feasible point satisfies $x_1 = v^\top y \pm \sqrt{y^\top V y + 1}$, we see that $x_1 = \pm 1$.

Continuity of Dual Optimal Solution

Theorem

Let x_θ be a critical point of (Q_θ). Let $s = \operatorname{codim}\mathcal{X}$, and let σ_θ be the s-th largest singular value of $D_g(x_\theta)$.

1. *If $ACQ_{\mathcal{X}}(x_\theta)$ holds, then there exists $\lambda_\theta \in \Lambda_\theta(x_\theta)$ with $\|\lambda_\theta\| \leq \frac{1}{\sigma_\theta}\|\nabla q_\theta(x_\theta)\|$.*
2. *If $ACQ_{\mathcal{X}}(\overline{x})$ holds and $x_\theta \to \overline{x}$ as $\theta \to \overline{\theta}$, then there exists $\lambda_\theta \in \Lambda_\theta(x_\theta)$ with $\lambda_\theta \to 0$.*

Proof.

1. If $ACQ_{\mathcal{X}}(x_\theta)$ holds, then $\operatorname{rank}(D_g(x_\theta)) = s$, so $\sigma_\theta > 0$. Since $\Lambda_\theta(x_\theta)$ is the solution space of $\lambda^\top D_g(x_\theta) = -\nabla q_\theta(x_\theta)$. The theorem follows from noticing that the 2-norm of the pseudo-inverse of $D_g(x_\theta)$ is $\frac{1}{\sigma_\theta}$.

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2. Since $\text{ACQ}_{\mathcal{X}}(\overline{x})$ must hold in a neighborhood of $\overline{x}$, and $\nabla q_\theta(x_\theta) = 2Q_\theta x_\theta = 0$, the result follows from the first part.

Proof of General Result

Recall the main theorem:

Theorem

Consider (Q_θ)*. Suppose* $Q(\theta)$ *is continuous, and* $c \neq 0$*. Suppose* $\overline{\theta}$ *is such that* $Q(\overline{\theta}) \succeq 0$ *has corank 1, and* $q^\star(\theta) = 0$*. If* $ACQ_{\mathcal{X}}(x)$ *holds, then* (Q_θ) *is SDP stable near* $\overline{\theta}$ *and* (P_θ) *shares its minimizer.*

Proof.

We have shown that $\overline{\lambda} = 0$ is an optimal solution for (D_θ), and we are given that $H_{\overline{\theta}}(0) = Q_{\overline{\theta}}$ has corank 1. Applying previous lemmas show that we have $(x_\theta, \lambda_\theta) \in \mathfrak{L}(\theta)$ such that $x_\theta \to \overline{x}$ and $\lambda_\theta \to 0$ as $\theta \to \overline{\theta}$. Then we have weak continuity, so applying another previous theorem completes the result.

Conclusion

- We have shown that a particular family of parametrized problems is robust under some weak assumptions
- The results also extend to the case where we can perturb the constraints
- We also are interested in more qualitative bounds under possibly more restrictive assumptions.

References I

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